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The almost-highest wave: a simple approximation

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The crest of a steep, symmetric gravity wave is shown to be closely approximated by the expression

$$x+iy = \frac{\alpha+\gamma i\chi}{(\beta+i\chi)^{\frac{1}{3}}},$$

where x, y are co-ordinates in the vertical plane, χ is the complex velocity potential and α, β, γ are certain constants. This expression is asymptotically correct both for small and for large values of $|\chi|$; and the free surface agrees with the exact profile calculated by Longuet-Higgins & Fox (1977) everywhere to within 1·5 per cent. The pressure at the surface is constant to within 5 per cent.

1. Introduction

In two recent papers (Longuet-Higgins & Fox 1977, 1978)† the asymptotic form of a steep, symmetric gravity wave near the wave crest was calculated precisely, and it was shown how to match the local flow analytically to the velocity field in the rest of the wave. The inner solution representing the motion near the crest was shown to be a self-similar flow, given by a series with coefficients which were calculated numerically.

This solution greatly simplified the calculation of the properties of steep gravity waves. But for some purposes it is convenient to have an even simpler approximation in which the inner flow itself is given by only a single term. It is the purpose of this note to find such an approximation. In fact we shall obtain an expression which reproduces the main features of the exact inner solution, and agrees with it to within a few percent over the entire range of application.

In papers to follow, this approximation will be used in the calculation of the orbits of the fluid particles near the crest of a steep wave, and in a study of the dynamical stability of such waves. But because of its own interest the derivation is given here separately.

2. The exact solution

As in paper I, consider a wave progressing horizontally to the left. Then in a reference frame moving with the phase-speed the motion appears as a steady flow to the right (see I, figure 2). We take axes with x directed vertically downwards and y horizontally to the right and set

$$\left.\begin{aligned} z &= x+iy = re^{i\theta}, \\ \chi &= \phi+i\psi, \\ w &= u+iv = (d\chi/dz)^*, \end{aligned}\right\} \tag{2.1}$$

and

† The first of these will be referred to as paper I.

where ϕ and ψ are the velocity potential and the stream function and u, v denote the components of velocity in the x and y directions. A star denotes the conjugate complex quantity.

We choose units of length and time so that

$$g = 1, \quad q^2 = 2, \tag{2.2}$$

where g denotes gravity and q denotes the particle speed at the wave crest.

The conditions to be satisfied by the inner solution are, first, that the pressure at the free surface must be constant; hence

$$(d\chi/dz)\,(d\chi/dz)^* = 2gx \quad \text{when} \quad \psi = 0. \tag{2.3}$$

Second, that the free surface pass through the point $z = 1$ and be symmetric about the line $y = 0$, hence

$$\chi = 0 \quad \text{when} \quad z = 1. \tag{2.4}$$

Third, for large values of $|z|$ the solution must tend asymptotically to the Stokes corner-flow:

$$i\chi \sim \tfrac{2}{3}z^{\frac{3}{2}} \quad \text{as} \quad |z| \to \infty. \tag{2.5}$$

In § 6 of I, the inner solution was expressed as an infinite series:

$$z = (\delta + i\chi)^{\frac{2}{3}}\,(b_0 + b_1\omega + b_1\omega^2 + \ldots), \tag{2.6}$$

where

$$\omega = \frac{\beta - i\chi}{\beta + i\chi} \tag{2.7}$$

and β, δ and $b_0, b_1, b_2, \ldots$ are real constants. It might seem natural, therefore, to approximate this solution by truncating the series after a finite number of terms. Retaining for example only the first two terms of the series we should have

$$z \doteqdot (\delta + i\chi)^{\frac{2}{3}}\,\frac{(b_0 + b_1)\,\beta + (b_0 - b_1)\,i\chi}{\beta + i\chi} = z_1(\chi), \tag{2.8}$$

say. This expression is further simplified if, as in the numerical calculations, we take $\delta = \beta$. However, an examination of the numerical values of the coefficients b_n, given in table 2 of I, shows that

$$z_1(0) = \delta^{\frac{2}{3}}(b_0 + b_1) = 2{\cdot}596 \tag{2.9}$$

not unity, as required by (2.4). The reason for this large discrepancy is to be found in the slow rate of convergence of the series in (2.6) when $\omega = 1$.

Nevertheless it will be seen that the form of (2.8) is asymptotically correct both when $|\chi| \ll 1$ and when $|\chi| \gg 1$, provided only that the constants in the expression are suitably chosen.

3. A simple approximation

We will therefore ignore the numerical values found in I and try instead

$$z = \frac{\alpha + \gamma i\chi}{(\beta + i\chi)^{\frac{1}{3}}}, \tag{3.1}$$

where α, β and γ are to be determined so as to satisfy the conditions of § 2, as far as possible. (α will not be confused with the quantity δ/β of paper I.)

To satisfy the condition (2.4) at $\chi = 0$ we must clearly have

$$\alpha = \beta^{\frac{1}{3}}. \tag{3.2}$$

Next, as $|\chi| \to \infty$ we have $z \sim \gamma(i\chi)^{\frac{2}{3}}$, so that to satisfy the condition (2.5) we must have

$$\gamma = (\tfrac{3}{2})^{\frac{2}{3}} = 1{\cdot}3104. \tag{3.3}$$

Thirdly, since

$$\frac{dz}{di\chi} = \frac{\gamma}{(\beta+i\chi)^{\frac{1}{3}}} - \frac{1}{3}\frac{\alpha+\gamma i\chi}{(\beta+i\chi)^{\frac{4}{3}}} \tag{3.4}$$

it follows that in order to satisfy (2.2) we must have

$$\frac{\gamma}{\beta^{\frac{1}{3}}} - \frac{1}{3\beta} = \frac{1}{2^{\frac{1}{2}}}. \tag{3.5}$$

Since γ is already determined, equation (3.5) may be regarded as a cubic equation to determine β. In fact writing $\beta = \lambda^{-3}$ we have

$$\lambda^3 - 3\gamma\lambda + 3\times 2^{-\frac{1}{2}} = 0, \tag{3.6}$$

which has roots

$$(\lambda_1, \lambda_2, \lambda_3) = \frac{12^{\frac{1}{3}}}{2^{\frac{1}{2}}} \times \left(1, \frac{3^{\frac{1}{2}}-1}{2}, \frac{-3^{\frac{1}{2}}-1}{2}\right). \tag{3.7}$$

The corresponding values of β are

$$\left.\begin{aligned} \beta_1 &= \frac{1}{3\times 2^{\frac{1}{2}}} = 0{\cdot}2357,\\ \beta_2 &= \frac{2^{\frac{3}{2}}}{3^{\frac{5}{2}}-15} = 4{\cdot}8065,\\ \text{and}\qquad \beta_3 &= \frac{-2^{\frac{3}{2}}}{3^{\frac{5}{2}}+15} = -0{\cdot}0925. \end{aligned}\right\} \tag{3.8}$$

Thus altogether we obtain

$$z = \frac{1+\epsilon i\chi}{(1+\beta^{-1}i\chi)^{\frac{1}{3}}}, \tag{3.9}$$

where

$$\epsilon = (\tfrac{3}{2})^{\frac{2}{3}}\beta^{-\frac{1}{3}} \tag{3.10}$$

and β takes one of the values (3.8).

4. Curvature of the wave crest

To see which of the values of β is the most suitable, consider the curvature of the profile of the free surface at the crest itself ($\chi = 0$). Generally, when $\psi = 0$ and $|\phi| \ll 1$ we have from (3.9)

$$x+iy = \frac{1+\epsilon i\phi}{(1+\beta^{-1}i\phi)^{\frac{1}{3}}} = 1 + Ai\phi + B\phi^2 + \ldots, \tag{4.1}$$

where

$$A = \epsilon - \frac{1}{3\beta} \quad\text{and}\quad B = \frac{\epsilon}{3\beta} - \frac{2}{9\beta^2}. \tag{4.2}$$

The radius of curvature R at $\phi = 0$ is given by

$$R = \lim_{\phi\to 0}\frac{y^2}{2(x-1)} = \frac{A^2}{2B}. \tag{4.3}$$

Corresponding to the three values of β in (3.8) we find respectively

$$\left.\begin{aligned} R_1 &= -\tfrac{1}{4} &&= -0{\cdot}250,\\ R_2 &= \quad(18\times 3^{\frac{1}{2}}-31)^{-1} &&= \quad 5{\cdot}652,\\ R_3 &= -(18\times 3^{\frac{1}{2}}+31)^{-1} &&= -0{\cdot}016. \end{aligned}\right\} \tag{4.4}$$

and

If the crest is to be convex, then R must be positive, so we choose $R = R_2$, $\beta = \beta_2$. The precise value of R determined in I was $R = 5{\cdot}15$.

5. Behaviour at infinity

When $|\chi|$ is large, we have from (3.9)

$$z = \frac{\gamma i\chi(1+\epsilon^{-1}/i\chi)}{(i\chi)^{\frac{1}{3}}(1+\beta/i\chi)^{\frac{1}{3}}} \tag{5.1}$$

$$= (\tfrac{3}{2}i\chi)^{\frac{2}{3}}\left(1+\frac{C}{i\chi}+O\left(\frac{1}{i\chi}\right)^2\right), \tag{5.2}$$

where

$$C = \epsilon^{-1}-\tfrac{1}{3}\beta = -0{\cdot}314. \tag{5.3}$$

Since C is negative, it follows that the profile approaches its asymptotes ($\theta = \pm\frac{1}{3}\pi$) from the *outside*, just as does the exact solution (see I, figure 9), though the latter has a damped oscillation at infinity. In (5.2), the angular departure $\Delta\theta$ of the free surface from its asymptote is given by

$$\Delta\theta \sim C\phi^{-1} \sim \tfrac{3}{2}Cr^{-\frac{3}{2}} = -0{\cdot}47r^{-\frac{3}{2}} \tag{5.4}$$

compared with the expression

$$\Delta\theta \sim 0{\cdot}78r^{-\frac{3}{2}}\cos(\tfrac{3}{2}\mu\ln r-\nu) \tag{5.5}$$

derived from equation (7.4) of I. We shall see that the difference between these two expressions is in practice small. Although (5.4) is monotonic and (5.5) is oscillatory, both quantities are negligible by the time that the asymptote is crossed a second time, namely at $r = 68{\cdot}5$ (see I, figure 9).

6. The complete profile

The entire profile of the surface is given by

$$x+iy = \frac{\alpha+\gamma i\phi}{(\beta+i\phi)^{\frac{1}{3}}}, \tag{6.1}$$

where $\alpha = 1{\cdot}6876$, $\beta = 4{\cdot}8065$ and $\gamma = 1{\cdot}3104$. Thus x and y are given as functions of the parameter ϕ. The result is shown in figure 1. It can be seen that the approximate expression (6.1) agrees with the exact profile remarkably well. The profiles each cross the asymptote at almost the same point, $r = 3{\cdot}32$. Out as far as $r = 45$ the maximum difference between the profiles $y(x)$ is less than $0{\cdot}02$ or about $1{\cdot}5$ per cent. For large values of r the angular difference in $\theta(r)$ for the two profiles is always less than $0{\cdot}01$, or about half a degree.

It is interesting to test the constancy of the pressure at the free surface by considering the ratio of the terms on the two sides of equation (2.3). At the crest $\phi = 0$

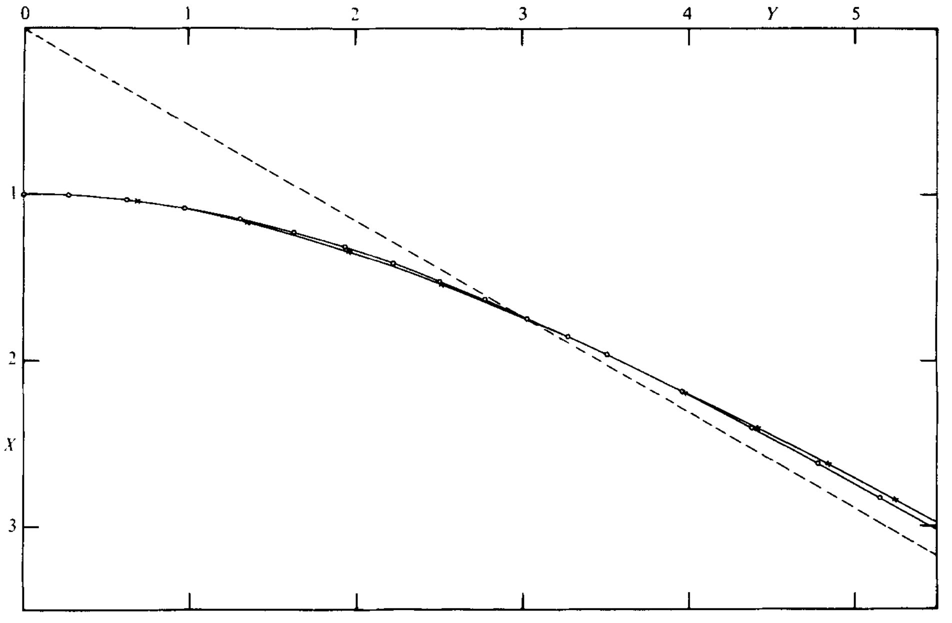

FIGURE 1. A comparison of the simple approximation (6.1) with the surface profile calculated accurately by Longuet-Higgins & Fox (1977). The broken line shows the asymptote, which is the same for both curves. ×, exact profile; ○, approximation (6.1).

and in the limit $\phi \rightarrow \infty$ the ratio equals 1 by arrangement. Over the rest of the profile, a calculation shows that the ratio never differs from 1 by more than 5 per cent, the maximum being 1·0498 when $y = 2{\cdot}72$. The ratio equals 1 again at $y \doteqdot 7{\cdot}07$ and thereafter never departs from 1 by more than 1 per cent. The pressure condition is therefore well satisfied.

7. Conclusions

The simple expression (6.1) gives a very good approximation to the asymptotic form of the almost-highest wave. This has been achieved by choosing the constants α, β, γ to fit the profile at the two *ends* of the range, that is at $\phi = 0$ and ∞, rather than at one end only. The approximation may be useful in calculating approximately the fluid velocities and particle orbits in a steep, symmetric wave, and in other problems.

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